REFLEXIVE DIGRAPHS WITH NEAR UNANIMITY POLYMORPHISMS

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In Celebration of the Seventieth Birthday of Ralph McKenzie

Abstract. In this paper we prove that if a finite reflexive digraph admits Gumm operations, then it also admits a near unanimity operation. This is a generalization of similar results obtained earlier for posets and symmetric reflexive digraphs by the second author and his collaborators. In the special case of reflexive digraphs our new result confirms a conjecture of Valeriote that states that any finite relational structure of finite signature that admits Gumm operations also admits an edge operation. We also prove that every finite reflexive digraph that admits a near unanimity operation admits totally symmetric idempotent operations of all arities. Finally, the aforementioned results yield a polynomial-time algorithm to decide whether a finite reflexive digraph admits a near unanimity operation.

1. Introduction

In order to discuss the problem we investigate we need to define some basic notions. Most of the definitions we use throughout the paper are quite standard in model theory and universal algebra and are covered in the texts [8] and [17]. An m*-ary relation* on a set A is a subset of A^m. An n*-ary operation* f on a set A is a map from A^n to A. Under a *unary, binary* or *ternary relation (operation)* we mean a 1-ary, 2-ary or 3-ary relation (operation), respectively.

A *signature* L is a set of symbols, each of which has a certain arity that is a non-negative integer. A *relational structure of signature* L is a non-empty set with a set of relations that are assigned to the elements of L such that each symbol and the relation assigned to it have the same arity. An *algebra of signature* L is a non-empty set with a set of operations that are assigned to the elements of L such that each symbol and the operation assigned to it have the same arity. If A is a relational structure (algebra) of signature L and r is a symbol of L, then r_A denotes the relation (operation) that is assigned to r.

Given two relational structures A and B of signature L, a map f from the underlying set of A to the underlying set of B is a *homomorphism* if for any symbol $r \in L$ whose arity is m and for any m-tuple $(a_1, \dots, a_m) \in r_A$ we have $(f(a_1), \dots, f(a_m)) \in r_B$. The *$n$-th power A^n of a relational structure A of signature L* is a relational structure of signature L such that the underlying set of A^n is the nth Cartesian power of the underlying set of A and for each symbol $r \in L$ of arity m the relation r_{A^n} consists of the m-tuples $((a_{1,1} \dots, a_{n,1}), \dots, (a_{1,m} \dots, a_{n,m}))$

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of the Cartesian power, where $(a_{1,1}\dots,a_{1,m}),\dots,(a_{n,1}\dots,a_{n,m})\in r_A$. An *n-ary polymorphism* of a relational structure A is a homomorphism A^n to A. Clearly, every polymorphism of A is an operation on the underlying set of A. A relational structure A *admits an operation* f if f is a polymorphism of A. A unary polymorphism is an *endomorphism.* A bijective endomorphism is an *automorphism.*

A *digraph* is a relational structure whose only relation is binary. In graph theory in this situation the underlying set of the structure is called the *vertex set* of the digraph and the relation is called the *adjacency relation* or the *set of edges.* Notice that by the above definition a homomorphism between two digraphs just means an edge-preserving map as usual in graph theory. A digraph is *reflexive, irreflexive, transitive, symmetric*, or *antisymmetric* if its adjacency relation has the same property. A *poset* is a reflexive, transitive and antisymmetric digraph.

Let now L be a signature of algebras. By using any finite set $\{x_1,\dots,x_n\}$ of variables and the symbols of L we define the *L-terms* recursively as follows: each of $x_1,\dots,x_n$ is an L-term; and if f is a k-ary symbol of L and $t_1,\dots,t_k$ are L-terms, then $f(t_1,\dots,t_k)$ is an L-term. In an algebra A of signature L the *L-terms interpret* as n-ary operations naturally: x_i interprets as the n-ary ith projection operation on the underlying set of the algebra for $1\le i\le n$; $f(t_1,\dots,t_k)$ interprets as the composition of f_A and the n-ary operations that are the interpretations of the terms $t_1,\dots,t_k$ in A. The operations obtained in this way are the *n-ary term operations* of the algebra. Now it makes sense to talk about whether an identity determined by two L-terms is satisfied by an algebra of signature L; if the term operations that correspond to the two terms are equal we say that *the identity is satisfied by the algebra.* Any class of algebras of signature L satisfying a set of identities of signature L is called a *variety.*

Near unanimity, congruence distributivity, and congruence modularity of varieties are well known properties for algebraists, see [17]. All of these properties can be characterized by an infinite sequence of finite sets of identities, called "Maltsev conditions"; for definition see [7]. The terms occurring in these Maltsev conditions are called a near unanimity term, Jónsson terms, and Gumm terms, respectively. The study of relational structures admitting the interpretations of these terms is of interest not just for a better understanding of algebraic structures, but for applications in the field of "constraint satisfaction problems", see [1],[2], [5], and [9].

We define the interpretations of the special terms listed in the preceding paragraph. An n-ary operation f is a *near unanimity operation* if $n\ge 3$ and f satisfies the identities

$$f(y,x,\dots,x)=f(x,y,x,\dots)=\dots=f(x,\dots,x,y)=x$$

in two variables x and y. A *majority operation* is a ternary near unanimity operation.

The ternary operations $d_0,\dots,d_n$ are *Jónsson operations* if they satisfy the identities

$$\begin{aligned} x&=d_0(x,y,z),\\ d_i(x,y,x)&=x \text{ for all } i,\\ d_i(x,y,y)&=d_{i+1}(x,y,y) \text{ for even } i,\\ d_i(x,x,y)&=d_{i+1}(x,x,y) \text{ for odd } i, \text{ and}\\ d_n(x,y,z)&=z. \end{aligned}$$

The ternary operations $d_0, \dots, d_n$, and p are *Gumm operations* if they satisfy the identities

$$\begin{aligned} x &= d_0(x,y,z), \\ d_i(x,y,x) &= x \text{ for all } i, \\ d_i(x,y,y) &= d_{i+1}(x,y,y) \text{ for even } i, \\ d_i(x,x,y) &= d_{i+1}(x,x,y) \text{ for odd } i, \\ d_n(x,y,y) &= p(x,y,y), \text{ and} \\ p(x,x,y) &= y. \end{aligned}$$

Without making it precise we mention that both sets of identities in the preceding two definitions encode certain connectivity properties in the set of ternary operations. It is well known and easy to prove that structures with a near unanimity polymorphism admit Jónsson operations, and structures with Jónsson polymorphisms obviously admit Gumm operations. It is also well known that in the class of all finite relational structures the converses of these implications fail to hold. It is natural to ask that for what kind of relational structures some of the three conditions coincide.

In a 1990 paper [16] Ralph McKenzie formulated the question of whether every finite poset that admits Jónsson operations also admits a near unanimity operation. The second author in his PhD thesis gave a positive answer to this question in the case of bounded posets and published the result in [19]. Larose and the second author extended this result in [14] by proving that if a finite poset admits Gumm operations, then it admits a near unanimity operation. Kun and Szabó gave a polynomial-time algorithm for testing whether a finite poset admits a near unanimity operation [11] and a polynomial-time algorithm for constructing Jónsson operations, provided the poset admits them [12]. In [13] Larose, Loten, and the second author proved that if a finite symmetric reflexive digraph admits Gumm operations, then it also admits a near unanimity operation and that these properties are decidable in polynomial time. On the basis of all of the results mentioned in this paragraph a general conjecture emerged: if a finite relational structure of finite signature admits Jónsson operations, then it admits a near unanimity operation. Recently in [1] Barto settled this conjecture in its full generality by using techniques he developed for studying the constraint satisfaction problem.

An n-ary operation f is an *edge operation* if $n \geq 3$ and f satisfies the identities

$$f(y,y,x,x,\dots,x) = f(x,y,y,x,\dots,x) = x$$

and

$$f(x,x,x,y,x,\dots,x) = f(x,x,x,x,y,x,\dots,x) = \cdots = f(x,x,x,\dots,x,y) = x$$

in two variables x and y. A *Maltsev operation* is a ternary edge operation f, that is, f satisfies the identities $f(y,y,x) = f(x,y,y) = x$. An operation f is *idempotent* if it satisfies the identity $f(x,\dots,x) = x$. All of the special operations we defined so far, such as the near unanimity, Jónsson, Gumm or edge operations are idempotent operations. Notice that adding three fictitious variables to a near unanimity operation at the beginning yields an edge operation. So if a relational structure admits a near unanimity operation, then it admits an edge operation. It is not hard to prove that if a relational structure admits an edge operation, then it admits Gumm operations. The converses of these implications over the class of all finite structures are well known to be false.

Nevertheless, a conjecture attributed to Valeriote states that if a finite relational structure of finite signature admits Gumm operations, then it admits an edge operation. The importance of the conjecture for the theory of constraint satisfaction problems is rooted in the following facts. For the finite relational structures of finite signature that admit an edge operation the constraint satisfaction problem is polynomial-time decidable by a generic algorithm, see [9]. For the finite relational structures of finite signature that admit Gumm operations no such a generic polynomial-time algorithm is known. If the conjecture was true, it would yield a generic polynomial-time algorithm for the constraint satisfaction problem over the broad class of structures that admit Gumm operations, a result that seems unreachable by direct methods at present.

In [15] Markovic and McKenzie proved that if a finite relational structure admits Jónsson operations and an edge operation, then it admits a near unanimity operation. Hence a positive answer to Valeriote's conjecture would yield a generalization of Barto's result. The main results in [13] and [14] also confirm the conjecture in the special cases of symmetric reflexive digraphs and posets. A further supporting evidence is a recent result of Kazda [10] that asserts that if a finite digraph admits a Maltsev operation, then it admits a majority operation.

In Section 2 we prove the main result of the paper: if a finite reflexive digraph admits Gumm operations, then it admits a near unanimity operation. This result also gives further support for Valeriote's conjecture, and extends the results of the second author and his collaborators mentioned above. In Section 3 we prove that every finite reflexive digraph that admits a near unanimity operation also admits totally symmetric idempotent operations of all arities. In Section 4, by the use of the results obtained in the earlier sections, we present a polynomial-time algorithm that decides whether a finite reflexive digraph admits a near unanimity operation or, equivalently, admits Jónsson or Gumm operations.

2. CM implies NU for reflexive digraphs

Our goal is to prove that if a finite reflexive digraph admits Gumm operations, then it admits a near unanimity operation. Actually, by Barto's result mentioned in the introduction we only have to prove that if a finite reflexive digraph admits Gumm operations, then it admits Jónsson operations. This is what we shall do in this section.

Let G be a digraph, and let a and b be two vertices of G. We write $a \to b$ to mean that (a, b) is an edge in G. Similarly, $a \leftarrow b$ means that (b, a) is an edge in G, and $a \leftrightarrow b$ means that both (a, b) and (b, a) are edges of G. The *nth power of a digraph* is a special case of the nth power of a relational structure. So for a digraph G and a positive integer n, G^n is the digraph whose vertex set is the n-th Cartesian power of the vertex set of G and whose adjacency relation is defined by $(a_1, \dots, a_n) \to (b_1, \dots, b_n)$ if and only if $a_i \to b_i$ for $1 \le i \le n$. We define another type of power of a digraph, where the exponent itself also is a digraph. Let G and H be two digraphs. In the introduction we noted that a homomorphism from H to G is just an edge-preserving map from H to G. Let G^H denote the digraph whose vertex set is the set of all homomorphisms from H to G and whose adjacency relation is defined as follows: $f \to g$ if and only if whenever $a \to b$ in H also $f(a) \to g(b)$ in G.

The following statement on composition of homomorphisms is a trivial but very useful tool for building new edges from existing ones in powers of digraphs. We frequently apply it in the later proofs with no explicit mention.

Lemma 2.1. *Let F, G, and H be finite digraphs. Let f_i and g_i be vertices in G^F for $1 \leq i \leq n$, and let f and g be vertices in H^{G^n}. If $f_i \to g_i$ in G^F for $1 \leq i \leq n$ and $f \to g$ in H^{G^n}, then $f(f_1, \dots, f_n) \to g(g_1, \dots, g_n)$ in H^F.* □

A unary operation r is a *retraction* if $r^2 = r$. The image of a retraction is a *retract.* We mainly use the following lemma in induction proofs on the size of a digraph.

Lemma 2.2. *For a finite digraph G if $\mathrm{id}_G \to f$ in G^G, where f is different from id_G, then there is a non-surjective retraction r in G^G such that $\mathrm{id}_G \to r$ in G^G. If, in addition, $f \to \mathrm{id}_G$ also holds, then r can be chosen such that $\mathrm{id}_G \leftrightarrow r$.*

Proof. First suppose that f is non-surjective. If $\mathrm{id}_G \to f$ in G^G, then for any n, $\mathrm{id}_G \to f^n$ in G^G. We choose n such that $(f^n)^2 = f^n$. Let $r = f^n$. Clearly, r is a non-surjective retraction in G^G, and $\mathrm{id}_G \to r$ in G^G. Moreover, if $\mathrm{id}_G \leftrightarrow f$, then $\mathrm{id}_G \leftrightarrow r$ in G^G.

Suppose now that f is surjective and $\mathrm{id}_G \to f$ in G^G. Let $B_1, \dots, B_k$ be a list of the orbits of f, and let $b_i \in B_i$ for $1 \leq i \leq k$. We define a unary operation r on the vertex set of G by setting $r(b) = b_i$ for every $b \in B_i$ and $1 \leq i \leq k$. Clearly, $r^2 = r$ and r is non-surjective. We prove that the map r is an endomorphism of G and $\mathrm{id}_G \leftrightarrow r$ in G^G. First, by using that f^{-1} is a finite power of f and $\mathrm{id}_G \to f$ in G^G we get that $\mathrm{id}_G \to f^{-1}$ in G. Then by $f \to f$ and $\mathrm{id}_G \to f^{-1}$ it follows that $f \to \mathrm{id}_G$ in G^G. Now, $\mathrm{id}_G \to f$ implies that $\mathrm{id}_G \to f^n$ in G^G for all n, and $f \to \mathrm{id}_G$ implies that $f^n \to \mathrm{id}_G$ in G^G for all n. Thus, $f^m \to f^n$ in G^G for all m and n. Therefore, if f has an edge from an orbit A of f to an orbit B of f, then every pair in $A \times B$ must be an edge of G. Thus r is an endomorphism and $\mathrm{id}_G \leftrightarrow r$ in G^G, which concludes the proof. □

A *path* in a digraph is a list of vertices $a_0, \dots, a_n$ such that $a_i \to a_{i-1}$ or $a_{i-1} \to a_i$ for $1 \leq i \leq n$. A path $a_0, \dots, a_n$ is a *directed path* if $a_{i-1} \to a_i$ for $1 \leq i \leq n$. A (directed) path from a vertex a to a vertex b is a (directed) path of the form $a_0, \dots, a_n$, where $a_0 = a$ and $a_n = b$. A digraph G is *connected* if for any vertices a and b of G there is a path from a to b. A digraph G is *strongly connected* if for any vertices a and b of G there is a directed path from a to b. The *connected* and *strong components* of a digraph are meant in the usual graph-theoretic sense. We require a stronger connectivity notion for digraphs. A path whose consecutive vertices are connected by edges in both directions is a *symmetric path.* A digraph G is *extremely connected* if for any any vertices a and b of G there is a symmetric path from a to b. Now we define an equivalence relation on the vertices of a digraph G: two vertices are equivalent if and only if they are connected by a symmetric path allowing the empty path. The blocks of this equivalence relation are the *extreme components* of G.

Let G be a relational structure, and let H be the digraph whose vertex set is the set of ternary polymorphisms of G and whose adjacency relation is defined by $f \to g$ if and only if $f(x, y, y) = g(x, y, y)$ or $f(x, x, y) = g(x, x, y)$. Let H_0 be the subgraph of H spanned by the vertices f with $f(x, y, x) = x$. We note that the existence of Jónsson polymorphisms of G is equivalent to the property that the first and third projections are connected by a path within H_0. The existence of Gumm polymorphisms of G is equivalent to the property that there is a path from the first projection to the third projection in H such that all vertices but possibly the vertex p before the last vertex of the path are in H_0 and, in addition, $p(x,x,y)=y$. From now on we concentrate on reflexive digraphs.

Lemma 2.3. *Let G be a finite reflexive digraph. If G^G is either connected, or strongly connected, or extremely connected, then for all finite digraphs H the digraph G^H has the same property.*

Proof. We prove the claim for the connected case; the proofs for the strongly connected and extremely connected cases are similar. Since G^G is connected and, by reflexivity, contains the constant maps, id_G is connected to some constant map by a path in G^G. Composing this path with any vertex f in G^H yields a path from f to a constant map in G^H. The resulting constant map is the same for all vertices f in G^H. Thus, G^H is connected. □

We require the following lemma on certain identities satisfied by finite reflexive digraphs. The lemma and its proof are closely related to Lemma 4.1 and its proof in [14].

Lemma 2.4. *If G is a finite connected, strongly connected, or extremely connected reflexive digraph such that $\{\mathrm{id}_G\}$ is a connected, strongly connected, or extremely connected component of G^G, respectively, then the following hold.*

(1) *If G admits a ternary operation d that satisfies the identity $d(x,y,y)=x$ on G, then d is the first projection on G.*
(2) *If for any proper retract R of G the digraph R^R has the same connectivity property that is considered for G and G admits a ternary operation d that satisfies the identities $d(x,y,x)=x$ and $d(x,x,y)=x$ on G, then d is the first projection on G.*

Proof. We prove the two claims in the strongly connected case. The proofs of the connected and extremely connected cases are similar.

For any vertices a and b of G we define the unary operation $d_{a,b}$ by $d_{a,b}(x)=d(x,a,b)$ for any vertex x in G. Notice that, because of the reflexivity of G, $d_{a,b}$ is an endomorphism of G for any vertices a and b of G. We start with the proof of the first claim of the lemma. Since G is reflexive and strongly connected, so is G^2. Since G^2 is strongly connected, for any vertices a and b of G there is a directed path from (a,a) to (a,b) in G^2, and hence there is a directed path from $d_{a,a}=\mathrm{id}_G$ to $d_{a,b}$ in G^G. Similarly, as G^2 is strongly connected, for any vertices a and b of G there is directed path from $d_{a,b}$ to id_G. Since $\{\mathrm{id}_G\}$ is a strong component of G^G, all vertices lying on these paths are equal to id_G. This yields $d(x,a,b)=x$ for any vertices x,a, and b of G, that is, d is the first projection on G.

Now we prove the second claim of the lemma. Let r be a non-surjective retraction in G^G. Such an r exists, since G^G contains the constant maps and G has at least two elements. We choose r such that its range R is maximal with respect to containment in G. Let a be a vertex in $G\setminus R$.

Let $g(x)=d(x,r(x),a)$. Clearly, g is in G^G, and fixes the elements in $R\cup\{a\}$ by the identities in the claim. An appropriate power of g is a retraction in G^G. Since this power has larger range than r, it must equal id_G. So g is an automorphism of G.

Since R^R is strongly connected, there is a directed path in G^G from r to a constant map b. This yields a directed path from g to $d_{b,a}$ in G^G. Since G^2 is strongly connected, for any vertices c and d in G there is a directed path from (b, a) to (c, d) in G^2, and hence there is a directed path from $d_{b,a}$ to $d_{c,d}$ in G^G. So for any vertices c and d in G there is a directed path from g to $d_{c,d}$. Similarly, as G is strongly connected, there is a directed path from $d_{c,d}$ to g, as well.

The facts that $\{\mathrm{id}_G\}$ is a strong component of G^G and g is an automorphism in G yield that $\{g\}$ must also be a strong component of G^G. Since for all vertices c and d in G the endomorphisms g and $d_{c,d}$ are in the same strong component of G^G, $g(x) = d(x, c, d)$ for all vertices x, c, and d in G. Therefore, for any vertex c in G we have $g(c) = d(c, c, c) = c$, and so $g = \mathrm{id}_G$ and d is the first projection. □

We would like to derive some interesting facts concerning the various connectivity conditions of finite reflexive digraphs that admit Gumm operations. The proof of the following theorem is closely related to the proofs of Lemma 4.1 and Theorem 4.2 in [14] for finite posets.

Theorem 2.5. *Let G be a finite reflexive digraph that admits Gumm operations. If G is either connected, or strongly connected, or extremely connected, then G^G has the same property.*

Proof. First, we prove the claim in the strongly connected case. Suppose to the contrary that the claim is not true for strongly connected digraphs, and let G be a counterexample of the smallest cardinality. Clearly, G has at least two elements. Since every proper retract of G admits Gumm operations, for every proper retract R of G the digraph R^R is strongly connected. Next, we prove that in G^G the vertex id_G does not have both incoming and outgoing non-loop edges.

Let us suppose to the contrary that id_G has both incoming and outgoing non-loop edges in G^G. Then by Lemma 2.2 there are non-surjective retractions r and s such that (r, id_G) and (id_G, s) are edges of G^G. Since the image of s is a proper retract of G, there is a directed path from s to a constant map of G^G. So there is a directed path from id_G to a constant map. Similarly, by using r, there is a directed path from a constant map to id_G.

For any vertex f in G^G, by composing f with each of these directed paths, we get a directed path from f to a constant map and a directed path from a constant map to f. Since G is strongly connected, any two constant maps are connected via a directed path in both directions. Thus, for any vertices f and g in G^G there is a directed path from f to g in G^G. This means that G^G is strongly connected, a contradiction. From what we just proved so far it follows that $\{\mathrm{id}_G\}$ is a strong component of G^G.

Now we finish the proof of the theorem in the strongly connected case. Let $d_0, \ldots, d_n$, and p be Gumm operations admitted by G. Thus, these operations satisfy all of the following identities on G:

$$\begin{aligned} x &= d_0(x, y, z), \\ d_i(x, y, x) &= x \text{ for all } i, \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \text{ for even } i, \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \text{ for odd } i, \\ d_n(x, y, y) &= p(x, y, y), \text{ and} \\ p(x, x, y) &= y. \end{aligned}$$

We now apply the two claims of the preceding lemma to G. By the third line of Gumm identities we have $x = d_1(x, y, y)$, so d_1 is the first projection. Then the second and fourth lines of Gumm identities yield $d_2(x, y, x) = x = d_2(x, x, y)$, so d_2 is also the first projection. Continuing in this fashion we obtain $p(x, y, z) = x$. However, $p(x, x, y) = y$ by the last line of Gumm identities, which contradicts the fact that G has at least two elements. This concludes the proof of the theorem in the strongly connected case.

The proof in the connected case is very similar to the proof of the strongly connected case. We set a connected reflexive digraph G to be a counterexample of the smallest cardinality. A similar argument as in the first part of the proof of the strongly connected case shows that $\{\mathrm{id}_G\}$ is a one element connected component of G^G. Then the rest of the above proof goes through word by word leaving out the words 'directed', 'strong', and 'strongly' from the text.

In the extremely connected case the proof is an analogue of the proofs of the connected and strongly connected cases. □

We remark without going into details that, by the use of Lemma 2.4, along the lines of the preceding proof, Theorem 2.5 extends to finite reflexive digraphs that admit *Hobby-McKenzie operations for omitting types* **1** *and* **5**. The definition of these operations is given in the statement of Theorem 9.8 in [7].

Theorem 2.6. *Let G be a finite reflexive digraph that admits Hobby-McKenzie operations for omitting types* **1** *and* **5**. *If G is either connected, or strongly connected, or extremely connected, then G^G has the same property.* □

We are able to draw stronger consequences of connectivity of reflexive digraphs that admit Gumm operations in the following theorem whose proof is closely related to the proof of part *(3) implies (4)* of Theorem 4.3 in [14]. Let $I_2(G)$ denote the digraph spanned by the idempotent operations in G^{G^2}.

Theorem 2.7. *Let G be a finite reflexive digraph that admits Gumm operations. If G is either connected, or strongly connected, or extremely connected, then $I_2(G)$ has the same property.*

Proof. Let $d_0, \dots, d_n$, and p be Gumm polymorphisms of G. So they satisfy the identities displayed in the preceding proof. We prove the claim in the strongly connected case. The proofs of the connected and extremely connected cases are similar, except that we replace directed paths by paths in the connected case, and by symmetric paths in the extremely connected case.

So let us assume that G is strongly connected. Let f and g be two polymorphisms in $I_2(G)$. By Theorem 2.5 and Lemma 2.3, G^{G^2} is strongly connected. Let $C \subseteq$

$I_2(G)$ be the strong component of f in $I_2(G)$. Since G is reflexive, every vertex $h \in C$ is connected to g in G^{G^2} via a path of the form

$$h = h_0 \to h_1 \leftarrow h_2 \to h_3 \leftarrow \dots\ h_k = g.$$

Choose such a path of minimum length. Next, we prove that $k \leq 1$, and thus $h \to g$ in G^{G^2}.

Notice that for any vertex of t in G^{G^2} and any $i \in \{1, \dots, n\}$, by the second line of Gumm identities, $d_i(h, t, g)$ is in $I_2(G)$. So if the operations $t_1, \dots, t_l$ form a directed path in G^{G^2}, then the operations $d_i(h, t_1, g), \dots, d_i(h, t_l, g)$ form a directed path in $I_2(G)$. Hence, as G^{G^2} is strongly connected, $d_i(h, h, g)$ and $d_i(h, g, g)$ are in the same strong component of $I_2(G)$ for all i. Therefore, by using the Gumm identities, the polymorphisms $h = d_0(h, g, g)$ and $d_n(h, g, g) = p(h, g, g)$ are in the same strong component of $I_2(G)$, namely, in C.

Put $m = \lfloor \frac{k+1}{2} \rfloor$, and if k is odd, then put $h_{k+1} = g$, as well. Thus we have a path of even length connecting $h_0 = h$ and $h_{2m} = g$. Now consider the path

$$p(h, g, g) = p(h_0, h_{2m}, g) \to p(h_1, h_{2m-1}, g) \leftarrow p(h_2, h_{2m-2}, g) \to \dots\ p(h_m, h_m, g) = g.$$

By the minimality of k we must have $m \geq k$, and consequently $k \leq 1$, that is, there exists an edge from h to g.

Since there is a directed path from f to h in C and $h \to g$, we have a directed path from f to g in $I_2(G)$. Notice that, by switching the roles of f and g, we also have a directed path from g to f in $I_2(G)$. This concludes the proof of the theorem. □

Let G and H be digraphs. A digraph K spanned by some elements of G^H is an *idempotent G-subalgebra* if the vertex set of K is closed under the idempotent polymorphisms of G. In detail, this means that for every idempotent polymorphism $f : G^n \to G$ and vertices $f_1, \dots, f_n$ in K the homomorphism g defined by $g(x) = f(f_1(x), \dots, f_n(x))$ for any vertex x of H also is a vertex of K.

Corollary 2.8. *Let G be a finite reflexive digraph that admits Gumm operations. If G is either connected, or strongly connected, or extremely connected, then every idempotent G-subalgebra has the same property.*

Proof. Let K be an idempotent G-subalgebra in G^H, and let f and g be any vertices of K. Assume that G is connected. Then there is a path connecting the two projections in $I_2(G)$ by the preceding theorem. Plugging (f, g) in the binary idempotent operations of this path yields a path from f to g in K. The proof for the strongly connected and extremely connected cases are similar. □

Strongly connected digraphs that admit Gumm operations have a stronger connectivity property.

Theorem 2.9. *If a finite strongly connected reflexive digraph G admits Gumm operations, then G is extremely connected.*

Proof. We first prove that G^G is extremely connected. In order to do this we show that there is a non-surjective retraction s connected to id_G by edges in both directions. We choose a non-identity element r in G^G such that there is an edge from id_G to r. Such an r exists by Theorem 2.5. Let

$$A = \{f : \ f \text{ is a vertex of } G^G \text{ and } \mathrm{id}_G \to f\}.$$

It is easy to see that A is an idempotent G-subalgebra in G^G. Therefore, by the preceding corollary A is strongly connected. Hence there is a directed path from r to id_G in A, so there is an edge from a non-identity element f of A to id_G. Hence by Lemma 2.2 there is a non-surjective retraction s such that $\mathrm{id}_G \leftrightarrow s$.

Now, we apply an induction on the size of G. Let $S = s(G)$. The size of S is less than the size of G, and S inherits the properties being strongly connected and admitting Gumm operations from G. So by the induction hypothesis, id_S is connected to a constant map via a symmetric path in S^S. Composing this path with s we get a symmetric path from s to a constant map in G^G. By using $\mathrm{id}_G \leftrightarrow s$ we get a symmetric path from id_G to a constant map. This implies that G^G is extremely connected.

Let a be a fixed vertex of G. Since G is reflexive, the map $f \mapsto f(a)$ is a surjective homomorphism from G^G to G. As G^G is extremely connected, so is its homomorphic image G. □

Let G be an arbitrary digraph. Let

$$\mathrm{Neq}(f, g) = \{a : \ a \text{ is a vertex of } G \text{ and } f(a) \neq g(a)\}.$$

An edge (f, g) in G^G is a *non-refinable edge* if there is no proper nonempty subset A of $\mathrm{Neq}(f, g)$ such that changing the value of g to f on A yields a map in G^G. An edge (a, b) is a *critical edge* of G with respect to an edge (f, g) of G^G if $(g(a), f(b))$ is not an edge in G. Notice that if (a, b) is a critical edge with respect to (f, g), then a and b are in $\mathrm{Neq}(f, g)$. In the following lemma we characterize the non-refinable edges of G^G via critical edges. This is the last ingredient we need in order to prove our main theorem.

Lemma 2.10. *Let G be a finite digraph. An edge (f, g) in G^G is non-refinable if and only if $\mathrm{Neq}(f, g)$ is strongly connected via critical edges with respect to (f, g) or is one element.*

Proof. Let N be the digraph whose vertex set is $\mathrm{Neq}(f, g)$ and whose edge set is the set of critical edges. First, suppose that N is not strongly connected and is not one element. Then let A be a strong component of N, such that there are no critical edges from $N \setminus A$ to A. Now, by changing the value of g to f on A we obtain a map in G^G.

Conversely, suppose that N is strongly connected. For any proper nonempty subset A of $\mathrm{Neq}(f, g)$ by changing the value of g to f on A we get map that is not in G^G. Indeed, by strong connectivity, there is a critical edge going into A that prevents the new map from being in G^G. □

We note that in the poset and symmetric reflexive digraph cases $\mathrm{Neq}(f, g)$ is a one element set for every non-refinable edge (f, g), for different reasons, though. In the poset case every strongly connected component of the poset is one element, and in the symmetric reflexive digraph case there do not exist critical edges at all. It is easy to come up with a reflexive digraph G that admits a majority operation, such that G^G has a non-refinable edge (f, g) and $\mathrm{Neq}(f, g)$ has at least two elements. Let G be the reflexive digraph on the vertex set $\{0, 1, 2\}$ whose adjacency relation is obtained from the full relation by removing the edges $(0,1)$ and $(0,2)$. Let $\mathbf{0}$ denote the constant 0 map. Then G admits a majority operation, the edge $(\mathrm{id}_G, \mathbf{0})$ is a non-refinable edge in G^G, and $\mathrm{Neq}(\mathrm{id}_G, \mathbf{0}) = \{1,2\}$.

Now we have all tools to prove our main theorem for reflexive digraphs. We obtained similar theorems for posets in [14] and for symmetric reflexive digraphs in [13]. In [12] Kun and Szabó gave a method to construct Jónsson terms for finite posets. Our proof incorporates some ideas from their paper. In the proof we also use Barto's result in [1] that states that if a finite relational structure admits Jónsson operations, then it admits a near unanimity operation.

Theorem 2.11. *For a finite reflexive digraph G the following are equivalent.*

1. *G admits a near unanimity operation.*
2. *G admits Jónsson operations.*
3. *G admits Gumm operations.*
4. *For every connected component B of G the maps id_{B^2} and $r:(x,y)\mapsto(y,y)$ are connected by a path in $B^{2^{B^2}}$ whose vertices fix the diagonal elements in B^2, and for every strong component C of G the maps id_{C^2} and r are connected by a symmetric path in a similar manner in $C^{2^{C^2}}$.*
5. *For every connected component B and strong component C of G the digraph $I_2(B)$ is connected and the digraph $I_2(C)$ is extremely connected.*
6. *For every connected component B and strong component C of G the idempotent B-subalgebras are connected and the idempotent C-subalgebras are extremely connected.*

Proof. Clearly, (1) $\Rightarrow$ (2). Moreover, (2) $\Rightarrow$ (1) is immediate from Barto's result in [1]. By the proof of Corollary 2.8, (5) $\Rightarrow$ (6). Conversely, (6) $\Rightarrow$ (5), since $I_2(B)$ is an idempotent B-subalgebra and $I_2(C)$ is an idempotent C-subalgebra.

To see that (4) $\Rightarrow$ (5) we take a path $(p_0,q_0),\dots,(p_m,q_m)$ from id_{B^2} to r in $B^{2^{B^2}}$ such that for any $x\in B$ and $1\le i\le m$, $(p_i(x,x),q_i(x,x))=(x,x)$. Then the p_i are idempotent, p_0 is the first projection, and p_m is the second projection. So $p_0,\dots,p_m$ is a path connecting the two projections in $I_2(B)$, and hence $I_2(B)$ is connected. A similar argument works for the strong component part of the claim. We prove that (5) $\Rightarrow$ (4) also holds. If $I_2(B)$ is connected, then there is path $p_0,\dots,p_m$ that connects the two projections in $I_2(B)$. So p_0 is the first projection, p_m is the second projection, and p_i is idempotent for $1\le i\le m$. Then the path $(p_0,p_m),(p_1,p_m),\dots,(p_m,p_m)$ connects id_{B^2} and r in $B^{2^{B^2}}$, and its vertices fix the diagonal elements in B^2. A similar argument works for the strong component part of the claim.

So far we have proved the equivalence of items (1) and (2) and the equivalence of items (4), (5), and (6). It is obvious that (2) $\Rightarrow$ (3), and by Corollary 2.8 we have that (3) $\Rightarrow$ (6). Thus, to finish the proof it suffices to prove that (6) $\Rightarrow$ (2).

Since (1) is equivalent to (2) and a finite digraph admits a near unanimity operation if and only if each of its connected components does, it suffices to prove that (6) $\Rightarrow$ (2) for the connected components of the digraph. So we assume that G is a finite connected reflexive digraph that satisfies the conditions given in (6) and prove that G admits Jónsson operations.

There is a fine point about the powers of reflexive digraphs that we use in the next paragraph and later in the proof. For a reflexive digraph F, a digraph H and a positive integer n the digraphs $(F^n)^H$ and F^{nH}, where nH denotes the digraph formed by n disjoint copies of H, are naturally isomorphic. Hence we may identify every idempotent F^n-subalgebra with an idempotent F-subalgebra.

Let

$$I = \{f \in {G^3}^{G^3} : \ f(a,b,a) = (a,b,a) \text{ for any vertices } a \text{ and } b \text{ in } G\}.$$

We define a map s in I by $s(a,b,c) = (c,b,c)$ for all vertices $a, b,$ and c in G. Since I is an idempotent G^3-subalgebra in ${G^3}^{G^3}$, it is also an idempotent G-subalgebra. So I is connected. Let P be a path of endomorphisms of G^3 from id_{G^3} to s in I. We may assume that in P the consecutive vertices are connected by non-refinable edges of ${G^3}^{G^3}$.

Let (f,g) be an edge of P such that $\mathrm{Neq}(f,g)$ contains the elements (a,a,b) and (c,d,d) for some vertices $a,b,c,$ and d in G. By the preceding lemma there is a strong component of G^3 that contains $\mathrm{Neq}(f,g)$. Since every strong component of G^3 is a product of strong components of G, there exist strong components C, C' and C'' of G such that $\mathrm{Neq}(f,g) \subseteq C \times C' \times C''$. Since $(a,a,b),\ (c,d,d) \in \mathrm{Neq}(f,g)$, hence $(a,a,b),\ (c,d,d) \in C \times C' \times C''$, and so $C = C' = C''$. Thus, $\mathrm{Neq}(f,g) \subseteq C^3$ for some strong component of C of G.

Since f and g map strong components into strong components and both maps are in I, $f(C^3) \subseteq C^3$ and $g(C^3) \subseteq C^3$. Let

$$J = \{h \in {C^3}^{C^3} : \ h = t|_{C^3} \text{ and } t|_{G^3 \setminus C^3} = f|_{G^3 \setminus C^3} \text{ for some vertex } t \text{ in } {G^3}^{G^3}\}.$$

Clearly, $f|_{C^3}$ and $g|_{C^3}$ are in J, and J is an idempotent C^3-subalgebra in ${C^3}^{C^3}$. So J is an idempotent C-subalgebra. Since C is a strong component of G, J is extremely connected. Hence $f|_{C^3}$ and $g|_{C^3}$ are connected by a symmetric path in J, and so f and g are connected by a symmetric path in I.

Now, by inserting symmetric paths between consecutive vertices of P if necessary, we get a new path connecting id_{G^3} and s in I such that for any two consecutive vertices f and g of this path either $\mathrm{Neq}(f,g)$ contains no element the form (a,a,b), or $\mathrm{Neq}(f,g)$ contains no element of the form (c,d,d), or $f \leftrightarrow g$ in I. In the latter case let A be the set of all elements of the form (a,a,b) from $\mathrm{Neq}(f,g)$, and let h be the map obtained by changing g to f on A. Since there are no critical edges with respect to (f,g), we have that h is in I. Clearly, $f \to h \to g$ is a path in I. This yields that id_{G^3} and s are connected by a path Q such that for any two consecutive vertices f and g of Q either $\mathrm{Neq}(f,g)$ has no element of the form (a,a,b), or $\mathrm{Neq}(f,g)$ has no element of the form (c,d,d). Then the first components of the 3-tuples of Q are polymorphisms that, with possible duplications, are witnessing the Jónsson identities on G. $\square$

3. NU implies TSI of all arities for reflexive digraphs

An n-ary operation f is a *cyclic operation* if it satisfies the identity

$$f(x_1, x_2, \dots, x_n) = f(x_2, \dots, x_n, x_1),$$

and f is a *totally symmetric operation* if

$$f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$

whenever $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$ viewed as sets rather than multisets. In this section we shall prove that every finite reflexive digraph with a near unanimity operation admits a totally symmetric idempotent operation for every arity.

We define dismantlability for a digraph G. An endomorphism r of G is a *one point elementary retraction* of G if r fixes all but one element of G and either $\mathrm{id}_G \to r$ or $r \to \mathrm{id}_G$. A digraph G is *dismantlable* if there is a list $G_0, \dots, G_n$ of digraphs such that $G_0 = G$, G_n is a singleton and for each $1 \leq i \leq n$ the digraph G_i is the image of G_{i-1} under some one point elementary retraction of G_{i-1}.

It is well known that near unanimity implies dismantlability for finite connected posets [14] and finite connected reflexive symmetric digraphs [13].

A *directed cycle* is a closed directed path. A reflexive digraph is *acyclic* if all of its directed cycles have a single vertex. Clearly, every poset is an acyclic reflexive digraph. In the following lemma we extend the above mentioned result from finite connected posets to finite connected acyclic reflexive digraphs.

Lemma 3.1. *Every finite connected acyclic reflexive digraph that admits a near unanimity operation is dismantlable.*

Proof. Let G be a finite connected acyclic reflexive digraph that admits a near unanimity operation. By using Theorem 2.5 there is a non-surjective retraction r such that $r \to \mathrm{id}_G$ or $\mathrm{id}_G \to r$ in G^G. We assume that $\mathrm{id}_G \to r$, the case, where $r \to \mathrm{id}_G$ is similar. We may also assume that the edge (id_G, r) is non-refinable. Since G is acyclic, by the use of Lemma 2.10, the set $\mathrm{Neq}(\mathrm{id}_G, r)$ is one element. This means that r is a one point elementary retraction. Now, the retract $r(G)$ inherits all relevant properties of G, and an induction on the size of G finishes the proof. □

We say that a digraph has the *fixed point property* if each of its endomorphisms has a fixed vertex. It is well known [18] that dismantlable posets have the fixed point property. The proof of this result works for acyclic reflexive digraphs as well. For the sake of completeness we include it in the paper.

Lemma 3.2. *Every dismantlable acyclic reflexive digraph has the fixed point property.*

Proof. The proof goes by an induction on the size of G. Let f be an endomorphism of G and r a one point elementary retraction of G that maps a into a different vertex a^*. Put $R = r(G)$. Now, by the induction hypothesis $rf|_R$ has a fixed vertex, say b. If $f(b) \neq a$, then b is a fixed vertex of f and we are done. If $f(b) = a$, then $rf(b) = a^*$. So $b = a^*$, and $f(a^*) = a$. Since r is one point elementary, there is an edge connecting a^* and a, say $a^* \to a$. Then $a = f(a^*) \to f(a)$, and so $f^n(a) \to f^{n+1}(a)$ for all n. Since G is finite and acyclic, there is an n such that $f^n(a) = f^{n+1}(a)$, and so $f^n(a)$ is a fixed vertex of f. □

By putting together the preceding two lemmas we get the following corollary.

Corollary 3.3. *Every finite connected acyclic reflexive digraph that admits a near unanimity operation has the fixed point property.* □

A *clique* of a digraph G is subset of vertices of G that spans a complete digraph in G. A digraph G has the *fixed clique property* if any endomorphism of G preserves some clique of G. The following lemma and its proof can be found in [6], see Theorem 2.65.

Lemma 3.4. *Every dismantlable symmetric reflexive digraph has the fixed clique property.* □

This lemma and the result mentioned in the third paragraph of this section yield the following corollary.

Corollary 3.5. *Every finite connected symmetric reflexive digraph that admits a near unanimity operation has the fixed clique property.* □

We prove that this corollary generalizes to finite connected reflexive digraphs. The *symmetric skeleton* of a digraph G is the digraph obtained from G by removing all edges (a, b) of G, where (b, a) is not an edge of G.

Lemma 3.6. *Every finite connected reflexive digraph that admits a near unanimity operation has the fixed clique property.*

Proof. Let G be a finite connected reflexive digraph that admits a near unanimity operation, and let f be any endomorphism of G. Let γ be the equivalence given by the strong connectivity property on G. Clearly, by reflexivity every polymorphism of G preserves γ.

By factoring G out with γ we get an acyclic connected reflexive digraph that also admits a near unanimity operation. Hence G/γ has the fixed point property, and so the endomorphism induced by f on G/γ has a fixed vertex. This means that f preserves some of the strong components of G. Let C be a strong component preserved by f, and let D be the symmetric skeleton of C. Let m be a near unanimity operation on G. Then m preserves C and the near unanimity operation $m|_C$ is admitted by C. So by Theorem 2.9, D is connected, moreover it admits the near unanimity operation $m|_C$ and also admits $f|_C$. By Corollary 3.5, D has the fixed clique property. So $f|_C$ preserves a clique of D, and hence f preserves a clique of G. □

This lemma gives a key to the proof of the following theorem.

Theorem 3.7. *Every finite reflexive digraph that admits a near unanimity operation admits cyclic idempotent operations of all arities.*

Proof. It suffices to prove the claim for connected digraphs. Let G be a finite connected reflexive digraph that admits a near unanimity operation. Let I be the digraph formed by the n-ary idempotent polymorphisms of G. As I is an idempotent G-subalgebra, by Corollary 2.8, I is connected. Clearly, I admits a near unanimity operation and is reflexive, hence by the previous lemma I has the fixed clique property.

We define a unary operation f on I by $f(t(x_1, x_2, \dots, x_n)) = t(x_2, \dots, x_n, x_1)$. Now f is clearly an endomorphism of I, so there is a clique C of I that is preserved by f. Let t be any map in C. Since C is closed under f, for all $0 \leq i \leq n-1$ the operations $t(x_{1+i}, x_{2+i}, \dots, x_{n+i})$ are in C, where the indices are meant modulo n. So all of these maps are connected by edges in both directions in I.

Now we define an n-ary operation s on G^n by

$$s(x_{1+i}, x_{2+i}, \dots, x_{n+i}) = t(x_1, x_2, \dots, x_n)$$

for all $0 \le i \le n-1$, where the n-tuple $(x_1, \ldots, x_n)$ runs through a set of representatives of the sets $\{(y_{1+i}, y_{2+i}, \ldots, y_{n+i}) \ : \ i \in \{0, 1, \ldots, n-1\}\}$, where $y_1, y_2, \ldots, y_n$ are any vertices of G. Clearly, s is a well defined cyclic idempotent operation on G and it is also a polymorphism, since for all $0 \le i \le n-1$ the maps $t(x_{1+i}, x_{2+i}, \ldots, x_{n+i})$ are connected by edges in both directions in I. □

A *G-colored digraph* is a pair (H, f), where H is a finite digraph and f is a partial map from H to G. A G-colored digraph (H, f) is *extendible* if f extends to a fully defined homomorphism from H to G. A *colored element* is an element in the domain of f. We say that g is a *homomorphism* from (H, f) to (H', f') if g is a homomorphism from H to H' and $f = f'g$. We say that a G-colored digraph *contains* an other if its vertex set, edge set, and partial coloring contains the vertex set, edge set, and partial coloring of the other, respectively.

A G-colored digraph (H, f) is a *G-obstruction* if (H, f) is non-extendible but any (H', f') properly contained in (H, f) is extendible. It is immediate from the definition that if G is reflexive, then the base digraph of any G-obstruction is a connected irreflexive digraph. The following theorem is a special case of Theorem 1.17 in [20].

Theorem 3.8. *Let G be any finite digraph and $n \ge 3$. Then G admits an n-ary near unanimity operation if and only if the number of colored elements in any G-obstruction is at most $n-1$.* □

We call an obstruction a *tree obstruction* if its underlying digraph is an oriented tree. The following characterization of digraphs admitting totally symmetric idempotent operations of all arities is also well known. It is a combination of Theorem 19 of Feder and Vardi in [5] and Theorem 1 of Dalmau and Pearson in [4] applied to the special case of digraphs augmented by all unary one element relations.

Theorem 3.9. *Let G be a finite digraph. Then G admits totally symmetric idempotent operations of all arities if and only if every G-obstruction is a homomorphic image of a tree obstruction.*

By using the preceding two characterizations we prove the main result of this section.

Theorem 3.10. *Every finite reflexive digraph that admits a near unanimity operation admits totally symmetric idempotent operation of all arities.*

Proof. It suffices to prove the claim in the connected case, hence we assume that G is a finite connected reflexive digraph that admits a near unanimity operation. Let (H, f) be an arbitrary G-obstruction. We are going to prove that (H, f) is a homomorphic image of a tree obstruction.

In a digraph P a closed path whose edges are pairwise different is called a closed trail. The reflexive closure of connectivity by closed trails is an equivalence relation on P that we denote by γ_P. The classes of γ_P are called the trail components of P.

Let (Q, g) be a G-obstruction that is a preimage of (H, f) with the property that the number of colored elements of (Q, g) is maximal and the maximum number of edges in a trail component of Q is minimal. Such a (Q, g) must exists, since G admits a near unanimity operation and so the number of colored elements in the

G-obstructions has an upper bound. We denote by a the number of colored elements of (Q, g) and by b the maximum number of edges in a trail component of Q .

By the maximality of a each of the colored elements of (Q, g) has degree one, for otherwise we could split the colored element to obtain a new G-obstruction that is a preimage of (Q, g) and has more colored elements than (Q, g). So each colored element of (Q, g) constitutes a one element trail component of (Q, g). Clearly, Q/γ_Q is a tree, and so the one element trail components containing the colored elements of Q are leaves of this tree. They are the only leaves. Indeed, (Q, g) is an obstruction and the relation of G is reflexive, hence there are no non-colored leaves of Q/γ_Q. We shall prove that the non-leaf vertices of Q/γ_Q are also singletons, that is, (Q, g) is a tree obstruction.

Suppose this is not true and let M be the set of trail components with the maximum number of edges in Q/γ_Q. Let B_0 be an element of M and $M_0 = M \setminus \{B_0\}$. We define $s(B)$, $B \in M_0$, to be the number of leaves $A \in Q/\gamma_Q$ such that the shortest path between B_0 and A contains B. We put $s_{Q,g}$ for the minimum of the sums $\sum_{B \in M_0} s(B)$, when B_0 ranges over M. We call $s_{Q,g}$ the split number of $s_{Q,g}$. Intuitively, the smaller the split number is, the bushier the M part of the tree Q/γ_Q looks.

We construct an obstruction (Q', g') such that (Q, g) is a homomorphic image of (Q', g'), the number of colored elements of (Q', g') is a, the maximum number of edges in a trail component of Q' equals b, and $s_{Q',g'}$ is strictly less than $s_{Q,g}$. Notice that this gives a contradiction, since by repeating the construction for (Q', g') and so on we arrive at an infinite strictly decreasing sequence of natural numbers, namely, at the sequence of the split numbers of the obstructions constructed.

Let $B_0 \in M$ such that $s_{Q,g} = \sum_{B \in M_0} s(B)$. Let us fix an edge (c, d) in B_0. Let (Q_0, g_0) be the colored digraph obtained from (Q, g) by deleting (c, d). For each $k \geq 2$ we define a G-colored digraph as follows. We take k disjoint copies of (Q_0, g_0) and denote them by (Q_j, g_j) for $1 \leq j \leq k$. The element corresponding to any $q \in Q$ in Q_j is denoted by q_j. We connect the (Q_j, g_j) by the edges of the form (c_j, d_{j+1}) for $1 \leq j \leq k-1$, and call the resulting colored digraph (R_k, r_k).

We claim that there exists a k such that (R_k, r_k) is non-extendible. Suppose that this is not true. Then there exist some l and a homomorphism extension $v : R_l \to G$ of r_l such that $v|_{Q_j} = v|_{Q_{j'}}$ for some $j + 1 < j' \leq l$. Let t be a $(j - j')$-ary cyclic idempotent operation admitted by G. We define a map $\alpha : Q \to G$ by

$$\alpha(q) = t(v(q_j), v(q_{j+1}), \dots, v(q_{j'-1})).$$

By using that $v|_{Q_j} = v|_{Q_{j'}}$ and t is cyclic we get that $\alpha(c) \to \alpha(d)$, and so it is clear that α is a homomorphism. Since t is idempotent, α is an extension of g. This contradicts the non-extendibility of (Q, g).

We choose k to be minimal such that (R_k, r_k) is non-extendible. Since (R_k, r_k) is non-extendible, it contains an obstruction (Q', g'). Then there exists a natural homomorphism from (Q', g') to (Q, g). Moreover, every homomorphism from (Q', g') to (Q, g) must be surjective, for otherwise (Q', g') would be extendible. Thus, (Q, g) is a homomorphic image of (Q', g'). Hence the number of colored elements of (Q', g') is larger than or equal to a. Then by the maximality of a the number of colored elements of (Q', g') equals a. Clearly, no trail component of Q' contains more edges than a trail component of Q with the maximum number of edges. So by the minimality of b the maximum number of edges in a trail component of Q' coincides with b.

Let M' be the set of the trail components of Q' with b edges. All it remains to prove is that for some trail component $C_0 \in M'$ the sum $\sum_{C \in M_0'} s(C)$ over $Q'/\gamma_{Q'}$, where $M_0' = M' \setminus \{C_0\}$is less than $s_{Q,g}$. By the definition of (Q', g') for each trail component C of Q' with b edges there is a trail component B of Q such that C equals one of the k copies of B. We call the trail components of Q' that contain the copies of elements of B_0 the remnants of B_0. Clearly, each of the remnants of B_0 has less than b edges. We choose $C_0 \in M'$ such that the shortest path between C_0 and the remnants of B_0 in $Q'/\gamma_{Q'}$ does not contain any element of M_0'. Now, let $B \in M_0$ and $C_1, \dots, C_l$ the list of full copies of B in M_0'. If none of the C_i equal C_0, then $\sum_{i=1}^{l} s(C_i) \leq s(B)$. If some of the C_i equals C_0, then $\sum_{i=1}^{l} s(C_i) < s(B)$. Finally, for each $B \in M_0$ whose full copy is not present in M' put the inequality $0 < s(B)$. By summing up the aforementioned inequalities over $B \in M_0$, we obtain that

$$\sum_{C \in M_0'} s(C) < \sum_{B \in M_0} s(B) = s_{Q,g},$$

which concludes the proof. □

Finally, we note that the theorem does not hold for irreflexive digraphs, as for the two element digraph $(\{0,1\}, \{(0,1),(1,0)\})$ admits no cyclic operations of even arities.

4. An algorithm deciding NU for reflexive digraphs

In this section we describe a polynomial-time algorithm that decides whether a finite reflexive digraph admits a near unanimity operation. First, we are going to prove that if a finite reflexive digraph G admits a sequence of Jónsson operations, then it admits one whose length is bounded by a polynomial of $|G|$. We require a lemma for the proof of this statement. For a subset R of a digraph G we denote by G_R the digraph spanned in G^G by the endomorphisms that fix every element in R.

Lemma 4.1. *Let G be a finite digraph and R a retract of G. If for some retraction r with range R there is a path from id_G to r in G_R, then for any retraction s with range R there is a path of length at most $2|G|$ from id_G to s in G_R. Moreover, if the path from id_G to r is symmetric, then so is the path from id_G to s.*

Proof. Let $\mathrm{id}_G = r_0, \dots, r_n$ be a path of shortest length from id_G to r_n in G_R, where r_n is a retraction with range R. By composing this path with itself by sufficiently many times we may assume that the r_i are retractions. Suppose that $r_i(P) \subseteq r_{i-1}(P)$ for $0 < i \leq j < n$. We shall construct a path of retractions $\mathrm{id}_G = s_0, s_1, \dots, s_n = r_n$ such that $s_i(P) \subseteq s_{i-1}(P)$ if $0 \leq i \leq j+1$. To get a path of this form just take the path $\mathrm{id}_G = r_0, r_1, \dots, r_j, r_j r_{j+1}, \dots, r_j r_n$ in G_R and compose with itself by sufficiently many times. Proceeding by induction, in this way we may assume that there is a path of retractions $\mathrm{id}_G = r_0, r_1, \dots, r_n$ from id_G to r_n in G_R such that $r_i(P) \subseteq r_{i-1}(P)$ for $0 < i \leq n$.

We claim that the sequence of ranges of the r_i is strictly decreasing. Suppose not, say the ranges of r_j and r_{j+1} are the same. Then $r_j = r_{j+1} r_j$, and the range of $r_n r_j$ equals R. So the length of the path $\mathrm{id}_G = r_0, r_1, \dots, r_j = r_{j+1} r_j, \dots, r_n r_j$

in G_R is one shorter than it is supposed to be, a contradiction. Since the sequence of the ranges of the r_i are strictly decreasing, $n \leq |G|$.

Let now s be any retraction in G_R with range R. Then

$$\mathrm{id}_G = r_0, r_1, \ldots, r_n = sr_n, sr_{n-1}, \ldots, s\mathrm{id}_G = s$$

is a path of length at most $2|G|$ from id_G to s in G_R.

The proof of the second statement of the lemma is similar. $\square$

The *diameter* of a finite digraph is the maximum length of the shortest paths connecting any two vertices of the digraph.

Corollary 4.2. *If G is a finite connected reflexive digraph that admits Gumm operations, then the diameter of every idempotent G-subalgebra is at most $2|G|^2$. If, in addition, G is strongly connected, then the diameter of the symmetric skeleton of every idempotent G-subalgebra is at most $2|G|^2$.*

Proof. We prove the first claim of the corollary, the proof of the second claim is similar. Let

$$I = \{f \in G^{2^{G^2}} : \ f(a,a) = (a,a) \text{ for every vertex } a \text{ in } G\}.$$

We define a retraction r in I by $r(a,b) = (b,b)$ for any vertices a and b in G. Since I is an idempotent G^2-subalgebra, by Corollary 2.8 there is a path of endomorphisms from id_{G^2} to r in I. Then by the preceding lemma there is a path of length at most $2|G|^2$ from id_{G^2} to r in I. Notice that the first components of the vertices of this path form a path P of length at most $2|G|^2$ connecting the two projections in $I_2(G)$.

Let K be an idempotent G-subalgebra in G^H, and let f and g be any vertices in K. Plugging (f,g) in the binary idempotent operations of P we get a path of length at most $2|G|^2$ from f to g in K, which concludes the proof. $\square$

Theorem 4.3. *If a finite reflexive digraph G admits a sequence of Jónsson operations, then it admits one whose length is at most $16|G|^7$.*

Proof. A closer look at the proof of Theorem 2.11 gives the required bound as follows. In the proof the Jónsson operations we constructed came from a path that connects id_G and a retraction s in an idempotent G-subalgebra. By the preceding corollary a path like this can be taken with length at most $2|G|^2$.

Then we had to refine the edges of this path by inserting paths of length at most $|G|^3$ between consecutive vertices to get a path with non-refinable edges. So the resulting path with non-refinable edges has length at most $2|G|^5$.

Then in the proof we inserted certain symmetric paths between consecutive vertices of the path with non-refinable edges. These symmetric paths came from symmetric paths of idempotent C-subalgebras, where C is a strongly connected component of G. So by the preceding corollary the length of each symmetric path inserted can be bounded by $2|G|^2$. Then the length of the path we obtain in our proof by inserting these symmetric paths is at most $4|G|^7$.

Proceeding further in the proof we inserted one endomorphism between certain consecutive elements of the path. Finally, we got our Jónsson operations by taking the first components of the members of the resulting path, possibly duplicating some of the first components. Consequently, the length of the sequence of Jónsson operations obtained at the end of the procedure can be bounded by $16|G|^7$. □

Theorem 4.4. *There exists a polynomial-time algorithm that decides whether a finite reflexive digraph admits a near unanimity operation.*

Proof. The main result of [1] states that a finite relational structure admits a near unanimity operation if and only if it admits Jónsson operations, therefore it is enough to decide of a reflexive digraph G whether it admits Jónsson operations $d_0, \dots, d_n$ for some n. By the preceding theorem we may assume that $n \leq 16|G|^7$. We may actually assume that $n = 16|G|^7$ by adding projections at the end of the list of Jónsson operations.

For a given finite reflexive digraph G we will construct a constraint satisfaction problem instance P of polynomial size in $|G|$ such that P has a solution if and only if G admits Jónsson operations of size $n = 16|G|^7$.

Let the variables of P be $x_{i,a,b,c}$, where $i \in \{0, \dots, n\}$ and a, b, and c are arbitrary vertices of G. For all i and $(a, b, c) \to (a', b', c')$ in G^3 we impose the binary constraints $x_{i,a,b,c} \to x_{i,a',b',c'}$. Thus for any solution of P the maps defined as $d_i(a, b, c) = x_{i,a,b,c}$ are homomorphisms from G^3 to G. Next, we add the unary singleton constraints $x_{i,a,b,a} = a$, $x_{0,a,b,c} = a$, and $x_{n,a,b,c} = c$ for any i and for any vertices a, b, and c of G. Finally, we add the equality constraint between $x_{i,a,b,b}$ and $x_{i+1,a,b,b}$ for even i, and between the $x_{i,a,b,b}$ and $x_{i+1,a,b,b}$ for odd i. The construction ensures that P has a solution if and only if G has a sequence of Jónsson operations of length n.

Next we present a polynomial-time algorithm that actually finds a solution of P whenever there exists one, i.e., our algorithm produces Jónsson operations, provided there exist such.

First we assume that G admits a near unanimity operation, and describe our algorithm in this case. By Theorem 3.10, G admits a totally symmetric idempotent operation of any arity. These operations are also admitted by the equality and the unary singleton relations. So P is an instance of a constraint satisfaction problem over a structure whose relations are at most binary and are preserved by a totally symmetric idempotent operation of any arity. Then, by Theorem 1 in [4], the $(1, 2)$-consistency algorithm decides in polynomial time whether P has a solution. In fact, our algorithm makes repeated use of the $(1, 2)$-consistency algorithm as follows.

Let $x_1, \dots, x_m$ be the list of variables of P. Put $P_0 = P$, and let $j \in \{1, \dots, m\}$. In the jth step our algorithm works on the instance P_{j-1} as follows. It adds a unary constraint $x_j = c$ for some vertex c of G to the constraints of P_{j-1}, and runs the $(1, 2)$-consistency algorithm for the extended instance whose relations are also preserved by totally symmetric idempotent operations of all arities. It repeats this procedure going through the vertices of G one by one until for some vertex c in G, when x_j is set to c, the output relations of the $(1, 2)$-consistency algorithm are not empty. We must have such a c in G, since there is a solution of P_{j-1}.

After c is found P_j is defined to be the output of the $(1, 2)$-algorithm run for the instance P_{j-1} augmented with the constraint $x_j = c$. Then the instance P_j must have a solution, since it has nonempty relations and is an output of the $(1, 2)$-algorithm that operated on an instance whose relations are preserved by totally symmetric idempotent operations for all arities. Clearly, the relations of P_j are also preserved by these operations. Thus, we see that by going through all variables and obtaining the values of x_j in the vertex set of G for each $j \in \{1, \ldots, m\}$, our procedure finds a solution of P in polynomial time.

On the other hand if G does not admit a near unanimity operation, then we can still run the above algorithm that will stop in polynomial time without producing a solution of P. Consequently, the above algorithm is a polynomial time procedure deciding whether G admits a near unanimity operation. $\square$

We remark that for producing a polynomial-time algorithm we do not really have to use Theorem 3.10 and the (1,2)-consistency algorithm. We only have to use the fact that there are some constants l and k such that the (l, k)-consistency algorithm works properly for constraint satisfaction instances when G admits Jónsson operations. For example, a result in [2] also guarantees us appropriate values of l and k, namely, $l = 2$ and $k = 3$.

We also note that by the use of any of conditions (4) and (5) in Theorem 2.11 one can design a polynomial-time algorithm based on the (1,2)-consistency algorithm as in the proof of Theorem 4.4 to test whether a finite reflexive digraph admits a near unanimity operation. This is a consequence of the fact that, by Lemma 4.1 and Corollary 4.2, there is a polynomial bound on the length of the paths in items (4) and (5).

5. Concluding remarks

In Theorem 2.11 we gave a characterization of finite reflexive digraphs with Gumm operations. It would be nice to get a similar characterization of finite reflexive binary structures with Gumm operations. We cannot expect that such structures admit a near unanimity operation or Jónsson operations. All we can hope for, according to Valeriote's conjecture is that those reflexive binary structures admit an edge operation. We remark that in [1] a conversion between finite structures of finite type and finite binary structures is delineated with the property of preserving Maltsev conditions and reflexivity. So a proof of Valeriote's conjecture for finite (reflexive) binary structures would also confirm the conjecture for finite (reflexive) structures.

Kazda's result mentioned in the introduction and Theorem 2.11 suggest that it may be true that if a finite digraph admits Gumm operations, then it admits a near unanimity operation. However, the truth is that there exist finite digraphs that admit Gumm operations, but no near unanimity operation. This was proved by Bulin, Delic, Jackson, and Niven in manuscript [3], which was made public during the editorial process of the present paper. The results in [3] also imply that it suffices to prove Valeriote's conjecture for finite digraphs.

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